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A Simplification of Gibson's Theorem on Discrete Operator Semigroups

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A bounded operator T on a Banach space generates a discrete semigroup $\{T^k\}$ $(k = 0, 1, 2,...)$ which is called uniformly bounded if $\| T^k \| \leqslant M$ $(k = 0, 1, 2,...)$. In [2] Gibson characterizes uniformly bounded discrete semigroups by means of a condition on the resolvent of T. The purpose of this note is to show that Gibson's condition can be simplified, and at the same time to provide an alternative (and perhaps more direct) method of proof. The motivation for considering this discrete result arises in part from its important analog for (C_0) semigroups, the Hille–Yosida–Feller–Miyadera–Phillips (HYFMP) theorem. It is hoped that the result will find application in areas (such as finite difference equations) which involve approximation of continuous processes by discrete ones.

Given a complex number z for which $1/z$ is in the resolvent set of the operator T, we define the resolvent R_z of T by $R_z = (I - zT)^{-1}$. In what follows we shall apply to the analytic function R_z elementary results for Banach space valued analytic functions [1, p. 224].

THEOREM.

$$\| T^k \| \leqslant M \ (k = 1, 2,...) \Leftrightarrow \|(R_z - I)^n \| \leqslant M \left(\frac{|z|}{1 - |z|}\right)^n \qquad (|z| < 1, n = 1, 2,...).$$

LEMMA. *Under either of the above conditions,*

$$(R_z - I)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} (zT)^{k+n} \qquad (|z| < 1, n = 1, 2,...).$$

Proof of Lemma. Expanding $(1 - z)^{-1} - 1 = z/(1 - z)$ in a power series about 0,

$$\frac{z}{1-z} = \sum_{k=0}^{\infty} z^{k+1} \qquad (|z| < 1),$$

and, similarly,

$$[(1-z)^{-1}-1]^n = \left(\frac{z}{1-z}\right)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} z^{n+k}$$
$$(|z| < 1, n = 2, 3,...).$$

Given that $\| T^k \| \leqslant M$ $(k = 1, 2,...)$, it then follows by standard completeness arguments that $\sum_{k=0}^{\infty} (zT)^{k+1}$ converges absolutely for $|z| < 1$ and that the operator represented by this series is precisely $R_z - I$. Similarly it follows that

$$(R_z - I)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} (zT)^{n+k} \qquad (|z| < 1, n = 2, 3,...).$$

Given, on the other hand, that $R_z - I$ exists $(|z| < 1)$, it follows immediately that $\sigma(T) \leqslant 1$ where $\sigma(T)$ is the spectral radius of T. Given any complex number w with $|w| < 1$,

$$1 > \sigma(wT) = \lim_{k\to\infty} \|(wT)^k\|^{1/k},$$

showing that $\|(wT)^k\| \to 0$ as $k \to \infty$ and in particular that $\|(wT)^k\|$ $(k = 1, 2,...)$ is uniformly bounded. Finally, given any $|z| < 1$, write $z = rw$ with $|r| < 1$, $|w| < 1$. Then $\|(wT)^k\|$ $(k = 1, 2,...)$ is uniformly bounded so that by the earlier results,

$$\sum_{k=0}^{\infty} \binom{n+k-1}{n-1} [r(wT)]^{n+k} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} (zT)^{n+k}$$

converges to $(R_z - I)^n$.

Proof of Theorem.

$\Rightarrow$:
$$\|(R_z - I)^n\| = \left\| \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} (zT)^{n+k} \right\|$$
$$\leqslant M \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} |z|^{n+k}$$
$$= M \left(\frac{|z|}{1-|z|}\right)^n,$$

where $\| T^k \| \leqslant M$ $(k = 1, 2,...)$ and the lemma have been used.

$\Leftarrow$: For any $r \cdot \ni \cdot 0 < r < 1$,

$$\binom{n-1}{n-1} T^n = \frac{1}{2\pi i} \int_{|z|=r} \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} (zT)^{n+k} z^{-(n+1)} \, dz$$

$$= \frac{1}{2\pi i} \int_{|z|=r} (R_z - I)^n z^{-(n+1)} \, dz \qquad (n = 1, 2, \ldots),$$

using complex integration of Banach space valued analytic functions and the lemma. Hence we have, taking norms and using the resolvent condition,

$$\| T^n \| \leqslant \frac{1}{2\pi} \cdot 2\pi r \cdot M \left(\frac{r}{1-r}\right)^n \cdot r^{-(n+1)} = \frac{M}{(1-r)^n}.$$

Letting $r \to 0$ we obtain $\| T^n \| \leqslant M$ $(n = 1, 2, \ldots)$.

Note that to prove $\| T^n \| \leqslant M$ for any given n requires only that

$$\|(R_z - I)^n\| \leqslant M \left(\frac{|z|}{1-|z|}\right)^n$$

for that n. The theorem is applicable for any $M \geqslant 0$, and for $M \leqslant 1$ we have:

COROLLARY. *With $M \leqslant 1$, $\| T^k \| \leqslant M$ $(k = 1, 2, \ldots)$ if and only if*

$$\|(R_z - I)\| \leqslant M \cdot \frac{|z|}{1-|z|} \qquad (|z| < 1).$$

Remarks. It is interesting to note that in this result for discrete semigroups the sufficiency of the resolvent condition is proved by a "continuous" argument, while in its continuous analog (the HYFMP theorem) the sufficiency is generally deduced from a discrete argument. There are certain ways in which the discrete result does not parallel its continuous analog. In particular, a major significance of the HYFMP theorem is that an operator satisfying a certain resolvent condition generates a (C_0) semigroup (which turns out to be uniformly bounded), while here existence of the discrete semigroup is immediate. Also the resolvent condition in the HYFMP theorem can be seen to involve only the resolvent itself rather than $R_z - I$. In this light, a natural resolvent condition to replace the one presented here (see discussion in Gibson's paper) would be

$$\| R_z^n \| \leqslant \frac{C}{(1-|z|)^n} \qquad (|z| < 1, n = 1, 2, \ldots) \text{ for } \textit{some } C > 0.$$

This condition is readily seen to be necessary for $\| T^k \| \leqslant M$ $(k = 1, 2,...)$ but it is an open question as to whether it is sufficient. It can be shown by arguments similar to those used here that this condition implies that

$$\| T^k \| = O(k^{1/2}).$$

References

1. N. Dunford and J. T. Schwartz, "Linear Operators," Part I, Interscience, New York, 1958.
2. A. C. Gibson, A discrete Hille–Yosida–Phillips theorem, *J. Math. Anal. Appl.* **39** (1972), 761–770.